

Sprague-Grundy theory in bounded arithmetic (Preliminary Draft)

Satoru Kuroda
Gunma Prefectural Women's University

Abstract

In this paper, we formalize Sprague-Grundy theory for combinatorial games in bounded arithmetic. We show that in the presence of Sprague-Grundy numbers, a fairly weak axioms capture PSPACE.

1 Introduction

Since the seminal paper by Bouton [1], combinatorial games have been paid much attention in various branches of mathematics. The observation in [1] is later generalized by Grundy [6] and Sprague [11] to form a powerful tool for finding winning strategies which is called Grundy number or Sprague-Grundy number.

Deciding the complexity of perfect information games is also a major problem in computational complexity theory. Many combinatorial games are related to space complexity such as PSPACE. For instance, Schaefer [8] proved that the game Node Kayles played on undirected graphs is complete for PSPACE. while some games have much weaker complexity such as P or LOGSPACE.

In this paper we show that with the aid of Sprague-Grundy number, a fairly weak theory of two-sort bounded arithmetic can capture PSPACE. More precisely, we introduce a function computing Sprague-Grundy number for Node Kayles together with strategy functions for both players using Sprague-Grundy number to the system V^0 and show that any alternating polynomial time machine can be simulated by a game of Node Kayles.

Specifically, for an alternating Turing machine M and an input X , we construct in V^0 an undirected graph $G(M, X)$ such that Alice has an winning strategy if and only if M accepts X . Since the strategy functions are polynomial time computable in Sprague-Grundy function, this result suggests that Sprague-Grundy number has such a strong computational power that manages search through polynomial space.

There are a number of literature concerning bounded arithmetic for PSPACE. Buss [2] in his seminal paper defined a second order theory U_2^1 whose provably total functions coincide with PSPACE. Later, Skelley [9] defined a three sort system W_1^1 for PSPACE. While these theories require higher order objects compared to theories for classes inside the polynomial hierarchy, Eguchi [4] defined a PSPACE theory Σ_0^B -ID by extending the two sort language by predicates which represent inductive definition for Σ_0^B definable relations. Our theory V_{NK} presented in this paper is considered as a minimal theory for PSPACE as it is contained in any of the above theory. We also remark that an application of

bounded arithmetic to combinatorial game theory is also given by Soltys and Wilson [10] who showed that strategy stealing argument can be formalized in W_1^1 and in turn proved that the game Chomp is in PSPACE.

We can alternatively formalize our theory with a stronger base theory such as PV while introducing Sprague-Grundy function only. However we do not follow such an approach since formalizing in weak theory such as V^0 enables us to construct theories for combinatorial games having weaker computational power. Among such games we are particularly interested in the game NIM whose computational complexity is around LOGSPACE but no completeness result is known so far. We remark that this choice of base theory forces us to give a slightly more complicated construction of the graph $G(M, X)$.

There is a rich theory of combinatorial games with a number of games and so we hope that our result gives a neat framework for logical analysis of combinatorial games.

This paper is organized as follows: in section 2 we define our theory V_{NK} by extending V^0 by functions computing winning strategies. In section 3, we show that V_{NK} actually computes winning strategies for Node Kayles. Section 4 is devoted to the proof of our main theorem. In particular, we construct a graph so that players winning strategies witness accepting or rejecting computations.

2 Formalizing combinatorial games

We will formalize the argument for combinatorial games in the language of two-sort bounded arithmetic.

We will assume familiarity with basic notions and properties of two-sort bounded arithmetic. For a detail, readers should consult with textbooks such as [3].

Let \mathcal{L}_A^2 be the two sort language of Cook-Nguyen [3]. Basically, upper case letters denote binary strings and lower case letters denote natural numbers. We also adopt an unusual notation that vector presentation of lower case letters such as \bar{z} also denote strings. For a language L we denote the Σ_0^B formulas in L by $\Sigma_0^B(L)$.

The theory V^0 has defining axioms for symbols in \mathcal{L}_A^2 together with the bit-comprehension axiom for Σ_0^B formulas. We use many properties of V^0 in this paper whose details can be found in [3].

For a string X and a number $i < |X|$, $X(i)$ denotes both the predicate that the i th bit of i is 1 and the i th bit of X itself. The sequence of numbers are coded by a string and we define the i th entry of a sequence X by $X[k]$ and the length of X by $Len(X)$. For two sequences X and Y , we denote the concatenation by $X * Y$. Strings are sometimes identified with a binary sequence as $P = \langle p_0, \dots, p_n \rangle$. Coding such sequences and proving basic properties of sequences can be done in V^0 .

The game we consider is known as Node Kayles which is played over undirected graphs. We code graphs by a two-dimensional array where we assume that any node has an edge to itself. Two-dimensional arrays represent directed graphs in general and undirected graphs

are given as a symmetric relation which is coded by symmetric matrices. So we define

$$\begin{aligned}
DGraph(G) &\Leftrightarrow \\
&\forall x \in G \exists u, v < |G| (x = \langle u, v \rangle) \wedge \\
&(\forall u < |G| (\exists v < |G| (\langle u, v \rangle \in G \vee \langle v, u \rangle \in G)) \rightarrow \langle u, u \rangle \in G). \\
UGraph(G) &\Leftrightarrow DGraph(G) \wedge \forall u, v < |G| (\langle u, v \rangle \in G \rightarrow \langle v, u \rangle \in G). \\
Node(G) = V_G &= \{u < |G| \mid \langle u, u \rangle \in G\}.
\end{aligned}$$

We define the game Node-Kayles over undirected graphs to be an impartial game played by two players Alice and Bob (Alice always moves first) starting from a graph G and in the move with the option G' which is a subgraph of G , the player chooses a node $x \in Node(G)$ and returns the subgraph G_x which is defined by

$$Node(G'_x) = \{y \in Node(G') : \langle x, y \rangle \notin E_{G'}\}$$

and

$$\langle y, z \rangle \in G'_x \Leftrightarrow y, z \in Node(G'_x) \wedge \langle y, z \rangle \in G'.$$

For a sequence $\bar{w} = \langle w_1, \dots, w_l \rangle$ we define $G_{\bar{w}}$ inductively as

$$G_{\emptyset} = G, \quad G_{\bar{w}*v} = \begin{cases} (G_{\bar{w}})_v & \text{if } v \in Node(G_{\bar{w}}) \\ G_{\bar{w}} & \text{otherwise.} \end{cases}$$

The first player unable to move loses. So a game over G is coded by a sequence $\bar{w} = \langle w_1, \dots, w_l \rangle$ such that $w_1 \in G$, $w_{i+1} \in G_{\langle w_1, \dots, w_i \rangle}$ for any $i < l$, $G_{w_{l-1}} \neq \emptyset$ and $G_{w_l} = \emptyset$. Alice wins in the game W over G if $Len(w) \bmod 2 = 1$ and otherwise Bob wins.

Proposition 1 *The function computing $G_{\bar{w}}$ from G and \bar{w} is Σ_0^B -definable in V^0 .*

(Proof). It is easy to see that $G_{\bar{w}}$ is definable by the formula

$$\begin{aligned}
\varphi(G, G', \bar{w}) &\Leftrightarrow \\
&\forall u, v < |V_G| (\langle u, v \rangle \in G' \leftrightarrow (\langle u, v \rangle \in G \wedge \neg \exists w_i \in \bar{w} (w = u))).
\end{aligned}$$

so that $\forall G, \bar{w} \exists! G' \varphi(G, G', \bar{w})$ is provable in V^0 . □.

Now we will define our base theory for combinatorial games. First we introduce functions $sg(G)$, $\tau(G)$, $\tau_A(\langle b_0, \dots, b_l \rangle, G)$ and $\tau_B(\langle a_0, \dots, a_l \rangle, G)$ with the following defining axioms:

$$\begin{aligned}
G = \emptyset &\rightarrow sg(G) = 0, \\
\neg UGraph(G) &\rightarrow sg(G) = \max\{x \in V_G\} + 1, \\
UGraph(G) \wedge G \neq \emptyset &\rightarrow sg(G) = \min\{k < |V_G| : \forall x \in V_G k \neq sg(G_x)\}.
\end{aligned}$$

$$\tau(G) = \begin{cases} \min\{v \in V_G : sg(G_v) = 0\} & \text{if such } v \text{ exists.} \\ \max\{v \in V_G\} + 1 & \text{otherwise.} \end{cases}$$

$$\begin{aligned}
\tau_A(\emptyset, G) &= \tau(G), \\
\tau_A(\langle b_0, \dots, b_{l+1} \rangle, G) &= \tau_A(\langle b_0, \dots, b_l \rangle, G) * \langle b_{l+1}, \tau(G_{\tau_A(\langle b_0, \dots, b_l \rangle, G) * b_{l+1}}) \rangle
\end{aligned}$$

$$\begin{aligned}
\tau_B(\emptyset, G) &= \emptyset, \\
\tau_B(\langle a_0, \dots, a_{l+1} \rangle, G) &= \tau_B(\langle a_0, \dots, a_l \rangle, G) * \langle a_{l+1}, \tau(G_{\tau_B(\langle a_0, \dots, a_l \rangle, G) * a_{l+1}}) \rangle.
\end{aligned}$$

Definition 1 Let \mathcal{L}_{NK} be the language L_A^2 extended by function symbols $sg(G)$ $\tau(G)$, $\tau_A(\langle b_0, \dots, b_l \rangle, G)$ and $\tau_B(\langle a_0, \dots, a_l \rangle, G)$. The \mathcal{L}_{NK} theory V_{NK} comprises the following axioms:

- defining axioms for symbols in \mathcal{L}_{NK}
- $\Sigma_0^B(\mathcal{L}_{NK})$ -COMP: $\exists X < a \forall y < a (X(a) \leftrightarrow \varphi(a))$,
where $\varphi(a) \in \Sigma_0^B(\mathcal{L}_{NK})$ which does not contain free occurrences of X .

Thus V_{NK} is V^0 in the extended language \mathcal{L}_{NK} .

Remark. We need only functions sg and τ_A in order to axiomatize the theory V_{NK} since other two functions are definable from these functions. For instance, τ_G can be defined from sg and τ_B can be defined by τ_A . However we add these two functions to the language to make argument simple.

The following fact is well-known.

Proposition 2 V_{NK} proves $\Sigma_0^B(\mathcal{L}_{NK})$ -IND:

$$\varphi(0) \wedge \forall x (\varphi(x) \rightarrow \varphi(x+1)) \rightarrow \forall x \varphi(x).$$

3 Winning strategies in Sprague-Grundy system

We show that strategy functions τ_A and τ_B actually computes winning game instances for Alice and Bob respectively.

Definition 2 Define formulas $AW S_{\tau_A}(G, l)$ and $BW S_{\tau_A}(G, l)$ as follows:

$$\begin{aligned} AW S_{\tau_A}(G) \Leftrightarrow \quad & \forall l \forall \langle b_0, \dots, b_l \rangle [(l = \lfloor |V_G|/2 \rfloor \wedge \forall i < l b_i \leq |V_G| + 1) \\ & \rightarrow \exists l_0 \leq l (\forall i < l_0 (b_i \in \text{Node}(G_{\tau_A}(\langle b_0, \dots, b_{i-1} \rangle, G)) \wedge \\ & \tau(G_{\tau_A}(\langle b_0, \dots, b_{i-1} \rangle, G)) \in \text{Node}(G_{\tau_A}(\langle b_0, \dots, b_{i-1} \rangle, G))) \\ & \wedge b_{l_0} \notin \text{Node}(G_{\tau_A}(\langle b_0, \dots, b_{l_0-1} \rangle, G)))] \end{aligned}$$

$$\begin{aligned} BW S_{\tau_B}(G) \Leftrightarrow \quad & \forall l \forall \langle a_0, \dots, a_l \rangle [(l = \lceil |V_G|/2 \rceil \wedge \forall i < l a_i \leq |V_G| + 1) \\ & \rightarrow \exists l_0 \leq l (\forall i < l_0 (a_i \in \text{Node}(G_{\tau_A}(\langle a_0, \dots, a_{i-1} \rangle, G)) \wedge \\ & \tau(G_{\tau_A}(\langle a_0, \dots, a_{i-1} \rangle, G)) \in \text{Node}(G_{\tau_A}(\langle a_0, \dots, a_{i-1} \rangle, G))) \\ & \wedge a_{l_0} \notin \text{Node}(G_{\tau_A}(\langle a_0, \dots, a_{l_0-1} \rangle, G)))] \end{aligned}$$

Theorem 1 V_{NK} proves that

$$\forall G \{ \text{Ugraph}(G) \rightarrow ((sg(G) \neq 0 \rightarrow AW S_{\tau_A}(G) \wedge (sg(G) = 0 \rightarrow BW S_{\tau_B}(G))) \}.$$

(Proof). We argue inside V_{NK} .

Suppose that $sg(G) \neq 0$ and let $\langle b_0, \dots, b_l \rangle$ be a list of nodes in G where $l = \lfloor |V_G|/2 \rfloor$. We show that

$$\forall i < l (\forall j \leq i b_j \in \text{Node}(G_{\tau_A}(\langle b_0, \dots, b_{j-1} \rangle, G)) \rightarrow sg(G_{\tau_A}(\langle b_0, \dots, b_i \rangle, G) = 0) \quad (*))$$

The proof proceeds by induction on i .

If $i = 0$ then $(*)$ trivially follows by the assumption.

Suppose by the inductive hypothesis that $(*)$ holds for $i \geq 0$ and assume that

$$b_{i+1} \in \text{Node}(G_{\tau_A(\langle b_0, \dots, b_i \rangle, G)}).$$

Since $sg(G_{\tau_A(\langle b_0, \dots, b_i \rangle, G)}) = 0$, it must be that

$$sg(G_{\tau_A(\langle b_0, \dots, b_i \rangle, G)} * b_{i+1}) \neq 0$$

and by the definition of τ , we have

$$sg(G_{\tau_A(\langle b_0, \dots, b_{i+1} \rangle, G)}) = 0.$$

So we have $(*)$ for $i + 1$.

We argue similarly for the case of $sg(G) = 0$ and by noting that $(*)$ is a Σ_0^B formula, the claim is obtained by Σ_0^B -IND in V_{NK} . \square

4 Sprague-Grundy system captures PSPACE

Now we are ready to show our main result; the theory V_{NK} captures *PSPACE*.

Theorem 2 *A function is Σ_1^B definable in V_{NK} if and only if it is in PSPACE.*

(Proof). It is easy to show that functions sg , τ , τ_A and τ_B can be computed in PSPACE. So the only if part can be proved using the standard witnessing argument. Actually the provably total functions of the universal conservative extension of V_{NK} is the AC^0 closure of functions sg and τ_A . So Herbrand theorem implies the witnessing. Thus the proof of if part is given is the rest of this section. \square

We will show that any polynomial time alternating Turing machine can be simulated by a game in V_{NK} . First recall that PSPACE is equal to APTIME (cf. Papadimitriou [7]). So we actually show that any polynomial-time alternating Turing machine can be simulated by a game of Node Kayles.

We assume some harmless simplifications on alternating Turing machines. Let M be an alternating Turing machine with time bound $p(|X|)$ on input X . where we assume that $p(n)$ is even for all n . We assume that all computation of M on input X terminates exactly at time $p(|X|)$. We also assume that the space bound of M is $p(|X|)$. Furthermore, we assume that M is binary branching. So we formalize the transition function as

$$\delta_M(k, q, a) = \langle q_k, a_k, m_k \rangle$$

where $k = 0, 1$, q and q_k are states of Q and $a, a_k \leq 2$, $m_k \in \{-1, 0, 1\}$. We abuse the notation and write

$$\delta_M(k, C, C') \Leftrightarrow C' \text{ is the next configuration of } C \text{ along the path } k.$$

The final assumption is that M computes in normal form in the sense that it first guesses the path $P = \langle p_1, \dots, p_{i_{p(n)}} \rangle$ in the computation tree and then start computing using P .

We show that polynomial time bounded alternating Turing machines can be simulated by Node Kayles provably in V_{NK} .

Let $C_{INIT}(M, X)$ denote the initial configuration of M on input X . For a binary string P , we denote by $C(P, M, X)$ the configuration of M reachable from $C_{INIT}(M, X)$ along the path P . The predicate $Accept(C, M)$ denotes that C is an accepting configuration of M . Note that all these functions and predicates are definable in V^0 . We also define

$$\begin{aligned} Comp(\langle C_0, \dots, C_{p(|X|)} \rangle, P, M, X) &\Leftrightarrow \\ C_0 &= C_{INIT}(M, X) \wedge \forall i < p(|X|) \delta_M((P)_i, C_i, C_{i+1}), \\ Acomp(\langle C_0, \dots, C_{p(|X|)} \rangle, P, M, X) &\Leftrightarrow \\ Comp(\langle C_0, \dots, C_{p(|X|)} \rangle, P, M, X) &\wedge Accept(C_{p(|X|)}, M, X), \\ Rcomp(\langle C_0, \dots, C_{p(|X|)} \rangle, P, M, X) &\Leftrightarrow \\ Comp(\langle C_0, \dots, C_{p(|X|)} \rangle, P, M, X) &\wedge \neg Accept(C_{p(|X|)}, M, X), \end{aligned}$$

Theorem 3 *There exist functions $G(M, X)$, $Comp_A(M, X, P)$, $Comp_R(M, X, P)$, $Path_A(M, X, P)$ and $Path_R(M, X, P)$ which are Σ_1^B definable in V_{NK} such that the following formulas are provable in V_{NK} .*

- (1). $\forall M, X UGraph(G(M, X))$,
- (2). $\forall M, X, P (|P| = p(|X|)/2 \rightarrow (Len(Path_A(M, X, P)) = 2Len(P) \wedge \forall k < Len(Path_A(M, X, P)) (Path_A(M, X, P)[2k+1] = P[k]))$,
- (3). $\forall M, X, P (|P| = p(|X|)/2 \rightarrow (Len(Path_R(M, X, P)) = 2Len(P) \wedge \forall k < Len(Path_R(M, X, P)) (Path_R(M, X, P)[2k] = P[k]))$,
- (4). $\forall M, X$
 $\{(sg(G(M, X)) \neq 0 \rightarrow \forall P (|P| = p(|X|)/2 \rightarrow$
 $Acomp(Comp_A(M, X, Path_A(M, X, P)), Path_A(G(M, X), P), M, X)))$
 $\wedge (sg(G(M, X)) = 0 \rightarrow \forall P (|P| = p(|X|)/2 \rightarrow$
 $Rcomp(Comp_R(M, X, Path_R(M, X, P)), Path_R(G(M, X), P), M, X)))\}$

First we sketch the outline of the proof.

Let M be an alternating Turing machine and X be an input. We construct two graphs $G_A(M, X)$ and $G_B(M, X)$ so that each legitimate game instance of either games corresponds to a computation of M on input X . Specifically, the first $p(|X|)$ moves of the game constitute a path P with $|P| = p(|X|)$ followed by a list of moves which establishes a computation of M along the path P , if players move correctly. We require that $G_A(M, X)$ and $G_R(M, X)$ satisfy that a game instance I is A-winning if and only if I corresponds to an accepting and rejecting computation of M on X along P respectively.

Once the graph is constructed, we can extract functions $Comp_A(G, P)$, $Comp_B(G, P)$, $Path_A(G, P)$ and $Path_B(G, P)$ using strategy functions τ_A and τ_B .

Now we present details of the proof.

The construction of $G_A(M, X)$ and $G_R(M, X)$ is similar to that for the graph simulating QBF games in [8]. Let $M = (Q, \Sigma, \delta, q_0, q_A)$ be an alternating Turing machine with $Q = \{q_0, \dots, q_m\}$, $\Sigma = \{0, 1, 2\}$ where 2 denotes the blank symbol and $q_A = q_1$. The transition function is given as $\delta(p, q, a) = \langle q_p, a_p, m_p \rangle$ where $p \in \{0, 1\}$, $q, q_p \in Q$ and $m_p \in \{-1, 0, 1\}$ whose intended meaning is that if the current state is q , the head reads the symbol a and the path p is chosen then the state changes to q_p , the tape content of the current head position is overwritten by a_p and the head moves by m_p .

Let $s = p(|X|)$ be the number of alternations of M on X , $l_0 = p(|X|) + 2$ be the length of the sequence coding configurations and $n_0 = 2(s + 1)l_0$. It turns out that $s + n_0$ is

equal to the number of total moves in A-winning legitimate game instances. We construct the graph of $G_A(M, X)$ and $G_R(M, X)$ with layers $P_i, A_{i,j}, B_{i,j}$, of legitimate nodes, Y_i of illegitimate nodes and C_A, C_B of constraints nodes so that in the i th round, the player must choose her or his move from i th legitimate layer. Nodes in each layers are given as follows:

- P -layers $P_i = \{p_{i,0}, p_{i,1}\}$ for $0 \leq i < s$ represent the choice of i th path in the computation.
- A -layers $A_{i,j}$ corresponds to computation by Alice after the path is decided by choices from P_0, \dots, P_{s-1} and consists of nodes as follows:

$$\begin{aligned} A_{i,j} &= \{a_{i,j,0}^T, a_{i,j,1}^T, a_{i,j,2}^T\}, \quad 0 \leq j < s \\ A_{i,s} &= \{a_{i,k}^H : 0 \leq k < s\} \\ A_{i,s+1} &= \{a_{i,r}^Q : 0 \leq r < |Q|\} \end{aligned}$$

The intended meaning is that if Alice chooses nodes $a_{i,0,i_0}^T, \dots, a_{i,s-1,i_{s-1}}^T, a_{i,k}^H$ and $a_{i,r}^Q$ then Alice's computation of the i th configuration is $C_i = \langle q_r, k, i_0, \dots, i_{s-1} \rangle$.

- B -layers $B_{i,j} = \{b_{i,j}\}$ which are intended for Bob's moves for $0 \leq i < s$ and $0 \leq j \leq s+1$ or $i = s$ and $0 \leq j \leq s$. Note that Bob's have no choice of moves for these rounds. Also note that the number of B -layers is one less than that of A -layers.

We list these layers in the order that players choose their moves as

$$P_0, \dots, P_{s-1}, A_{0,0}, B_{0,0}, \dots, A_{s,s+1}, B_{s,s}.$$

So we sometimes denote layers by ignoring their types as

$$L_k = \begin{cases} P_k & \text{if } 0 \leq k < s, \\ A_{i,j} & \text{if } k = s + 2(i \cdot l_0 + j), \quad 0 \leq i \leq s, \quad 0 \leq j \leq s+1, \\ B_{i,j} & \text{if } k = s + 2(i \cdot l_0 + j) + 1, \quad 0 \leq i \leq s, \quad 0 \leq j \leq s. \end{cases}$$

We define constraint layers C_A and C_R for $G_A(M, X)$ and $G_R(M, X)$ respectively which expresses constraints for the computation of M . Nodes of these layers are labelled by propositional formulas and we identify nodes with their labels. The layer C_A and C_R contain the following nodes:

- (A) Nodes of the first sort are called initial nodes and express the initial configuration of M on X which consists of $\rightarrow a_{0,0}^Q, \rightarrow a_{0,0}^H$, for $j < |X|$, $\rightarrow a_{0,j,k}^T$ where $k = X(j)$ and for $|X| \leq j < s$, $\rightarrow a_{0,j,2}^T$.
- (B) The second sort are called transition nodes of M which consists of rules expressing the transition function of M . Specifically, let $c \in \{0,1\}$, $0 \leq j \leq m$, $z \in \{0,1,2\}$ and $\delta(c, q_j, z) = \langle q_{j'}, z', d \rangle$ for some $0 \leq j \leq |Q|$, $z' \in \{0,1,2\}$ and $d \in \{-1,0,1\}$. Then for $0 \leq i < s$ $0 \leq j \leq |Q|$ and $0 \leq k < s$, we introduce the following rules:

$$\begin{aligned} p_{i,c} \wedge a_{i,j}^Q \wedge a_{i,k}^H \wedge a_{i,k,z}^T &\rightarrow a_{i+1,k,z'}^T \\ p_{i,c} \wedge a_{i,j}^Q \wedge a_{i,k}^H \wedge a_{i,k',a}^T &\rightarrow a_{i+1,k',a}^T, \quad k' \neq k \\ p_{i,c} \wedge a_{i,j}^Q \wedge a_{i,k}^H \wedge a_{i,k,a}^T &\rightarrow a_{i+1,k+d}^H \\ p_{i,c} \wedge a_{i,j}^Q \wedge a_{i,k}^H \wedge a_{i,k,a}^T &\rightarrow a_{i+1,j'}^Q \end{aligned}$$

Note that these rules compute the $i + 1$ st configuration from the i th configuration which is specified by choosing the path c . We call a rule containing $p_{i,c}$ for $c = 0, 1$ as i -rule.

Moreover, C_A contains a single accepting node denoted by Acc while C_R contains a single rejecting node denoted by Rej .

Finally, the non-legitimate nodes are defined as

$$Y_{n_0-k} = \{y_{n_0-k, n_0-k+j} : 0 \leq j < k+1\}.$$

for $1 \leq k < n_0$.

Next we define edges among the nodes. In the following, let C denote either C_A or C_R .

1. For $0 \leq i < s$ and $c \in \{0, 1\}$, $p_{i,c} \in P_i$ is connected to all nodes in C which contains $p_{i,1-c}$.
2. For $0 \leq i \leq s$ and $0 \leq j \leq s+1$, $a \in A_{i,j}$ is connected to all nodes in C which either contain a in the succedent or $b \in A_{i,j}$ with $b \neq a$ in the antecedent.
3. The node $a_{s,1}^Q$ in $G_A(M, X)$ is connected to the node Acc .
4. The node $a_{s,j}^Q$ for $j \neq 1$ in $G_A(M, X)$ is connected to the node Rej .
5. all nodes in C are mutually connected.
6. All nodes in $L_k \cup Y_k$ for $1 \leq k \leq t_0$ are mutually connected.
7. The node $y_{t_0-k, t_0-k+j} \in Y_{t_0,k}$ is connected to all nodes in

$$\bigcup \{L_i \cup Y_i : t_0 - k < i \leq t_0 + 1, i \neq t_0 - k + j\}.$$

Proposition 3 *The function computing $G_A(M, X)$ and $G_R(M, X)$ from M and X is Σ_1^B definable in V^0 .*

(Proof). We code $G(M, X)$ in such a way that indices of nodes represent their labels. For instance, the node $p_{i,c}$ in P_i for $0 \leq i < s$ and $c \in \{0, 1\}$ is indexed by the tuple $\langle 0, i, c \rangle$ where the first entry 0 represents that it belongs to a P -layer.

Similarly, the node $a_{i,j}^Q$ in $A_{i,s+1}$ for $0 \leq i \leq s$ and $0 \leq j \leq |Q|$ is indexed by the tuple $\langle 0, s + i \cdot n_0 + 1, j \rangle$ and nodes in other A -layers and B -layers are indexed as well.

The node y_{n_0-k, n_0-k+j} in the layer Y_{n_0-k} is indexed by the tuple $\langle 1, n_0 - k, j \rangle$ for $0 \leq j < k+1$.

Finally nodes in $C_A \cup C_R$ are indexed by tuples of the form $\langle 0, n_0, t \rangle$ where t is a tuple coding its label. For instance the node

$$p_{i,c} \wedge a_{i,j}^Q \wedge a_{i,k}^H \wedge a_{i,h,a}^T \rightarrow a_{i+1,jc,a}^Q$$

is denoted by the tuple $\langle 0, i, c, j, k, a, 0, j_{c,a} \rangle$.

Then it is easy to see that the edge relation of $G(M, X)$ is definable by a Σ_0^B formula so it is defined by Σ_0^B -COMP. \square

We say that a subgraph G' of $G = G_A(M, X)$ or $G_R(M, X)$ is k -legitimate for $0 \leq k \leq s + n_0$ if

$$\text{Leg}(G', G, k) \Leftrightarrow \forall x \in V_G \left(\left(x \in \bigcup_{k' < k} L_{k'} \rightarrow x \notin V_{G'} \right) \wedge \left(x \in \bigcup_{k < k' \leq s+n_0} L_{k'} \rightarrow x \in V_{G'} \right) \right).$$

In the following, we denote $G = G_A$ or G_R if there is no fear of confusion.

The following lemma states that the graph $G(M, X)$ is constructed so that players are forced to choose their moves from legitimate nodes for otherwise they lead to an immediate loose.

Lemma 1 V_{NK} proves that from any legitimate graph G' of G , the first non-legitimate move leads to an immediate lose for either player:

$$\forall G' \forall < n_0 + s \forall x ((\text{Leg}(G', G, k) \wedge x \notin L_{k+1}) \rightarrow \text{sg}(G'_x) \neq 0).$$

(Proof). We argue in V_{NK} to show that if G' is a k -legitimate subgraph of $G(M, X)$ and $v \notin L_k$ then $\text{sg}(G'_v) \neq 0$.

Let $v \notin L_k$. Then either $v \in Y_j$ for $j \geq k$ or $v \in L_j$ for $j > k$. In the first case, we have $v = y_{j,l}$ for some l and taking it from G' removes all nodes except $L_l \cup Y_l$. Since $L_l \cup Y_l$ forms a complete subgraph, it must be that $\text{sg}(G'_{j,l}) \neq 0$.

In the second case, G'_v consists of all nodes in $L_l \cup Y_l$ with $l \neq j$ and nodes in L_{N_0+1} which are not connected to v . By the construction of $G(M, X)$, $y_{k,j}$ remains in G'_v and is connected to all nodes in G'_v . So we have $G'_{\langle v, y_{k,j} \rangle} = \emptyset$. This implies that $\text{sg}(G'_v) \neq 0$ as required. \square

We say that a sequence $\bar{w} = \langle v_1, \dots, v_m \rangle$ of nodes in $G_A(M, X)$ or $G_R(M, X)$ is legitimate, denoted by $SLeg(w, G)$, if $v_i \in L_i$ for all $i \leq m$. Then the following is an immediate consequence of Lemma 1.

Corollary 1 V_{NK} proves that

$$\forall k < t_0 \forall \langle v_1, \dots, v_k \rangle : \text{legitimate} \forall v_{k+1} (v_{k+1} \notin L_{k+1} \rightarrow \text{sg}(G_{v_1 \dots v_{k+1}}) \neq 0).$$

(Proof). It remains to show that if $\langle v_1, \dots, v_m \rangle$ is legitimate then for any $k \leq m$, $G_{v_1 \dots v_k}$ is a k -legitimate subgraph of $G(M, X)$ which can be proved by Σ_0^B -IND on $k \leq m$. \square

If both players move legitimately, The first s moves will be $p_{0,c_0}, \dots, p_{s-1,c_{s-1}}$ which decides the path $P = \langle c_0, \dots, c_{s-1} \rangle$ in the computation tree of M on X .

We require that if $\text{sg}(G_A(M, X)_P) = 0$ then Bob can win the game for $G_A(M, X)_P$ only if he moves consistently with the computation of M on X along the path P . Otherwise if $\text{sg}(G_A(M, X)_P) \neq 0$ then Alice can win the game for $G_R(M, X)_P$ only if she moves consistently with the computation along P .

In order to prove the above property of $G(M, X)$ in V_{NK} , we next show that each list of legitimate moves forms a list of configurations.

Note that we can divide A-layers and B-layers into consecutive lists $A_{i,0}, \dots, A_{i,s+1}$ and $B_{i,0}, \dots, B_{i,s+1}$. We call these two lists as the i -round. We assert that each set of

legitimate move by both Alice and Bob for the i -round forms a configuration of M on input X . Specifically, let Alice's moves for the i -round be given as

$$\bar{a}_i = a_{i,j}^Q, a_{i,k}^H, a_{i,0,a_0}^T, \dots, a_{i,s-1,a_{s-1}}^T.$$

Then we define $\text{conf}(\bar{a}_i) = \langle j, k, a_0, \dots, a_{s-1} \rangle$. Thus a legitimate sequence $\langle \bar{a}_0, \dots, \bar{a}_s \rangle$ of moves by Alice forms a sequence of configurations $\langle \text{conf}(\bar{a}_0), \dots, \text{conf}(\bar{a}_s) \rangle$.

We define legitimate moves by Alice and Bob after $s+2$ rounds as

$$\begin{aligned} \text{Leg}(\langle v_1, \dots, v_k \rangle, M, X) &\Leftrightarrow \forall j < k (v_{j+1} \in L_{s+1+i}), \\ A\text{-Leg}(\langle a_0, \dots, a_k \rangle, M, X) &\Leftrightarrow \forall j < k (a_{j+1} \in L_{s+2j}), \\ B\text{-Leg}(\langle b_0, \dots, b_k \rangle, M, X) &\Leftrightarrow \forall j < k (b_{j+1} \in L_{s+2j+1}). \end{aligned}$$

We omit parameters M and X if it is clear from the context. We also denote legitimate sequences of Alice and Bob as $\langle a_{0,0}, a_{0,1}, \dots, a_{i,j} \rangle$ and $\langle b_{0,0}, b_{0,1}, \dots, b_{i,j} \rangle$ respectively for $i \leq s$ and $j \leq s+1$.

Finally we define predicates which states that a given legitimate move form a computation of M .

$$\begin{aligned} \text{Comp}(\langle a_{0,0}, \dots, a_{s,s+1} \rangle, M, X, P) &\Leftrightarrow \\ \text{Leg}(\bar{a}, M, X) \wedge \text{conf}(\bar{a}_0) &= C_{\text{INIT}}(M, X) \wedge \forall i < s \delta_M(P(i), \text{conf}(\bar{a}_i), \text{conf}(\bar{a}_{i+1})), \\ \text{AComp}(\langle a_{0,0}, \dots, a_{s,s+1} \rangle, M, X, P) &\Leftrightarrow \\ \text{Comp}(\langle a_{0,0}, \dots, a_{s,s+1} \rangle, M, X, P) \wedge \text{Accept}(\bar{a}_s, M, X), \\ \text{RComp}(\langle a_{0,0}, \dots, a_{s,s+1} \rangle, M, X, P) &\Leftrightarrow \\ \text{Comp}(\langle a_{0,0}, \dots, a_{s,s+1} \rangle, M, X, P) \wedge \neg \text{Accept}(\bar{a}_s, M, X). \end{aligned}$$

Note that Bob's moves after s rounds are unique if he moves legitimately. So we denote $\bar{b} = \langle b_{0,0}, \dots, b_{s,s} \rangle$.

In the followings, M and X always denote a code of an alternating TM and its input respectively and we refrain from stating it explicitly.

For a sequence $X = \langle x_0, \dots, x_l \rangle$, We define the function $ASeq(X) = \{x_i : i \bmod 2 = 0\}$. Note that if X codes a game instance then $ASeq(X)$ gives a list of Alice's moves.

The next lemma states that the value of $sg(G_A(M, X)_P)$ for $|P| = s$ decides whether M accepts X along the path P .

Lemma 2 V_{NK} proves that

$$\forall M, X, P \left\{ |P| = s \rightarrow \begin{aligned} (sg(G_A(M, X)_P) \neq 0 \rightarrow \text{AComp}(ASeq(\tau_A(\langle b_{0,0}, \dots, b_{s,s} \rangle, G_A(M, X)_P), M, X, P))) \wedge \\ (sg(G_A(M, X)_P) = 0 \rightarrow \text{RComp}(ASeq(\tau_A(\langle b_{0,0}, \dots, b_{s,s} \rangle, G_R(M, X)_P), M, X, P))) \end{aligned} \right\}.$$

In order to prove Lemma 2, we first prepare some notations. As stated above, legitimate moves \bar{a}_i by Alice in a_i rounds is presented as

$$\bar{a}_i = a_{i,0,k_0}^T, \dots, a_{i,s-1,k_{s-1}}^T, a_{i,k}^H, a_{i,j}^Q$$

where $0 \leq j \leq m$, $0 \leq k \leq s-1$ and $k_0, \dots, k_{s-1} \in \{0, 1, 2\}$. Likewise, Bob's moves for a_i rounds is represented as $\bar{b}_i = b_{i,1}, b_{i,2}, \dots, b_{i,s+2}$ for $0 \leq i < s$ and $\bar{b}_s = b_{s,1}, b_{s,2}, \dots, b_{s,s+1}$.

We denote the moves by Alice and Bob for $G(M, X)_P$ respectively as

$$\bar{a} = \langle \bar{a}_0, \dots, \bar{a}_s \rangle \text{ and } \bar{b} = \langle \bar{b}_0, \dots, \bar{b}_s \rangle$$

We sometimes ignore the type of the nodes of Alice's move and denote by $a_{i,j}$ the j -th move of Alice in the i -round. Furthermore we define

$$\bar{a}^{\leq i,j} = \bar{a}_1 \dots, \bar{a}_{i-1}, a_{i,1}, \dots, a_{i,j} \text{ and } \bar{a}^{< i,j} = \bar{a}_1 \dots, \bar{a}_{i-1}, a_{i,1}, \dots, a_{i,j-1}.$$

$$\bar{a}^{\leq i} = \bar{a}_1 \dots, \bar{a}_i \text{ and } \bar{a}^{< i} = \bar{a}_1 \dots, \bar{a}_{i-1}.$$

The sequences $\bar{b}^{\leq i,j}$, $\bar{b}^{< i,j}$, $\bar{b}^{\leq i}$ and $\bar{b}^{< i}$ are defined similarly.

For sequences $\bar{a} = \langle a_0, \dots, a_k \rangle$ and $\bar{b} = \langle b_0, \dots, b_k \rangle$ or $\langle b_0, \dots, b_{k-1} \rangle$, we define the V^0 -definable function

$$\text{merge}(\bar{a}, \bar{b}) = \langle a_0, b_0, \dots, a_k, b_k \rangle \text{ or } \langle a_0, b_0, \dots, a_k, b_k \rangle.$$

respectively.

The proof of Lemma 2 is divided into a series of sublemmas. Define Σ_0^B formulas $\text{Init}(r, z, M, X)$ and $\text{Next}(r, z, p, C, M)$ so that

$$\begin{aligned} \text{Init}(r, z, M, X) &\Leftrightarrow z \text{ is the } r\text{th element of } C_{\text{INIT}}(M, X), \\ \text{Next}(r, z, p, C, M) &\Leftrightarrow z \text{ is the } r\text{th element of } C' \text{ with } \delta_M(p, C, C'). \end{aligned}$$

A $A_{i,j}$ -rule is a transition rule in C_A whose succedent contains a node in $A_{i,j}$. We say that a legitimate subgraph G' of $G(M, X)$ contains no $A_{i,j}$ -rule if there is no node in G' which belongs to C_A and represents some $A_{i,j}$ -rule. We also say that G' contains no A -rules if for all $i \leq s$ and $j \leq s+1$, G' contains no $A_{i,j}$ -rules. Note that these properties are formalized by a Σ_0^B formula.

Let $\bar{z} = \langle z_0, \dots, z_k \rangle$ be a list of legitimate moves by Alice or Bob for $k \leq (s+1)(s+2)$. We define that \bar{z} is a partial computation as

$$\begin{aligned} P\text{Comp}(\bar{a}, P, M, X) &\Leftrightarrow \\ A\text{-Leg}(\bar{a}, G_P) \wedge \forall k \leq \text{Len}(\bar{a}) \{ (q_k = 0 \rightarrow \text{Init}(r_k, a_k, M, X)) \wedge \\ (q_k > 0 \rightarrow \text{Next}(r_k, a_k, P(q_k - 1), \text{conf}(\bar{a}_{q_k-1}), M)) \}, \end{aligned}$$

where q_k and r_k are such that $k = q_k(s+1) + r_k$ and $0 \leq r_k \leq s+1$.

The next lemma states that moves by Alice or Bob must be consistent with the computation of M in order to obtain legitimate options.

Lemma 3 *Let G be either $G_A(M, X)$ or $G_R(M, X)$. Then V_{NK} proves that*

$$\begin{aligned} \forall M, X \forall P \forall l \leq (s+1)l_0 \forall \bar{a} = \langle a_0, \dots, a_l \rangle \forall \bar{b} = \langle b_0, \dots, b_l \rangle \\ \left\{ (|P| = s \wedge A\text{-Leg}(\bar{a}) \wedge B\text{-Leg}(\bar{b})) \wedge \text{Len}(\bar{a}) = \text{Len}(\bar{b}) + 1 \rightarrow \right. \\ \left. (P\text{Comp}(\bar{a}, P, M, X) \leftrightarrow \forall k \leq l (G_{P * \text{merge}(\bar{a}, \bar{b})} \text{ contains no } A_{q_k, r_k}\text{-rule})) \right\}. \end{aligned}$$

(Proof). We prove the claim of the lemma for $A_{i,j}$ -rules by induction on l . If $l = 0$ then we have to do nothing. So suppose that $l \geq 0$ and by the inductive hypothesis assume that the claim holds for l . Let us denote the lefthand side of the subformula inside the brace $\{\dots\}$ of the claim by $(*)_l$. Assume that $(*)_{l+1}$ holds, that is

$$\forall k \leq l+1 ((q_k = 0 \rightarrow \text{Init}(r_k, a_k, M, X)) \wedge (q_k > 0 \rightarrow \delta_M(p_{q_k-1}, \text{conf}(\bar{a}_{q_k-1}), 2r_k - 1, a_k)).$$

By the inductive hypothesis we already have

$$\forall k \leq l((G_P)_{\text{merge}(\bar{a}, \bar{b})} \text{ contains no } A_{q_k, r_k}\text{-rules}).$$

So it suffice to show that $(G_P)_{\text{merge}(\bar{a}, \bar{b})}$ contains no $A_{q_{l+1}, r_{l+1}}$ -rules

If $q_{l+1} = 0$ then we have $\text{Init}(r_{l+1}, a_{l+1}, M, X)$ and since $\rightarrow a_{l+1}$ is the only L_{l+1} -rule, we have the claim. Otherwise, we have

$$\text{Next}(2r_{l+1} - 1, a_{l+1}, p_{q_{l+1}-1}, \text{conf}(\bar{a}_{q_{l+1}-1}), M)$$

so there must be a rule in C of the form $A \rightarrow a_{l+1}$ where A represents a conjunction which is consistent with $\text{conf}(\bar{a}_{q_{l+1}-1})$. Furthermore, it is the only $A_{q_{l+1}, r_{l+1}}$ -rule which is in $(G_P)_{\text{merge}(\bar{a}_{\leq l}, \bar{b}_{\leq l})}$. Thus again we have the claim.

Conversely, suppose that $(*)_{l+1}$ does not hold. If $(*)_l$ does not hold then we have the claim by the inductive hypothesis. So suppose that

$$\begin{aligned} & (q_{l+1} = 0 \wedge \neg \text{Init}(r_{l+1}, a_{l+1}, M, X)) \\ & \vee (q_{l+1} > 0 \wedge \neg \text{Next}(2r_{l+1} - 1, a_{l+1}, p_{q_{l+1}-1}, \text{conf}(\bar{a}_{q_{l+1}-1}), M)). \end{aligned}$$

If the first disjunct is true then there exists an initial rule $\rightarrow y_{l+1}$ where $y_{l+1} \in L_{l+1}$ and $y_{l+1} \neq a_{l+1}$ which is not eliminated by the move a_{l+1} of Alice.

Otherwise if the second conjunct is true then we may assume that $(G_P)_{\text{merge}(\bar{a}, \bar{b})}$ does not contain any L_k -rule for $k \leq l$. Since

$$\neg \text{Next}(2r_{l+1}, y_{l+1}, p_{q_{l+1}-1}, \text{conf}(\bar{a}_{q_{l+1}-1}), M)$$

there must be a rule of the form $A \rightarrow y_{l+1}$ such that A is consistent with $\text{conf}(\bar{a}_{q_{l+1}-1})$ and so it remains in $(G_P)_{\text{merge}(\bar{a}, \bar{b})}$. Since $A \rightarrow y_{l+1}$ is not eliminated by a_{l+1} we have the claim. \square

Corollary 2 *Let G be either $G_A(M, X)$ or $G_R(M, X)$. Then V_{NK} proves that if Alice moves legitimately on G_P then she removes all A -rules if and only if her moves are consistent with the computation of M on X along P :*

$$\begin{aligned} & \forall M, X, P \forall \bar{a} \left\{ (|P| = s \wedge \text{Leg}(\bar{a}) \wedge \text{Len}(\bar{a}) = n_0) \rightarrow \right. \\ & \left. (\text{Comp}(\bar{a}, P, M, X) \leftrightarrow (G_{P * \langle e_0, e \rangle * \text{merge}(\bar{a}, \bar{b})} \text{ contains no } A\text{-rules of } M)) \right\} \end{aligned}$$

(Proof). We argue inside V_{NK} . First we remark that

- the move $a_{i,j}$ by Alice removes all nodes in C which contain $a_{i,j}$ in the succedent or $a' \in X_{i,j}$ with $a' \neq a_{i,j}$ in the antecedent and

- any move in \bar{b} by Bob does not remove any node in C .

We say that a node in C is a i -round rule if it is a $A_{i,j}$ -rule for some $0 \leq j \leq s+1$. We will prove that

$$\begin{aligned} \text{conf}(\bar{a}_0) = C_{INIT}(M, X) &\rightarrow \forall k \leq N \{ \forall i \leq k \delta_M(P(i), \text{conf}(\bar{a}_i), \text{conf}(\bar{a}_{i+1})) \\ &\leftrightarrow \forall i \leq k (G_P)_{\text{merge}(\bar{a} \leq i, \bar{b} \leq i)} \text{ contains no } i\text{-round rules} \}. \end{aligned}$$

The proof is by induction on k . For $k = 0$ we show that

$$\begin{aligned} \text{conf}(\bar{a}_0) &= C_{INIT}(M, X) \\ &\leftrightarrow (G_P)_{\text{merge}(\bar{a}_0, \bar{b}_0)} \text{ contains no initial rule of } M. \end{aligned}$$

Suppose first that $\text{conf}(\bar{a}_0) = C_{INIT}(M, X)$. Then each move $a_{0,j}$ of Alice removes the initial rule $\rightarrow a_{0,j}$ in L_{N_0} . Such a rule exists since $\text{conf}(\bar{a}_0) = C_{INIT}(M, X)$.

Conversely, suppose that $\text{conf}(\bar{a}_0) \neq C_{INIT}(M, X)$. Then for some choice $a_{0,j}$ of Alice, L_{N_0} contains the initial rule $\rightarrow z'_{0,j}$ with $a_{0,j} \neq z'_{0,j}$. Since $\rightarrow z'_{0,j}$ cannot be removed by any other moves in a_0 -rounds, $(G_P)_{\text{merge}(\bar{a}_0, \bar{b}_0)}$ must contain it.

For induction step, suppose that for $k \leq s-1$

$$\begin{aligned} (\text{conf}(\bar{a}_0) = C_{INIT}(M, X) \wedge \forall i < k \delta_M(P(i), \text{conf}(\bar{a}_i), \text{conf}(\bar{a}_{i+1}))) \\ \leftrightarrow \forall i < k (G_P)_{\text{merge}(\bar{a} \leq k, \bar{b} \leq k)} \text{ contains no } i\text{-round rules}. \end{aligned}$$

and we show that

$$\delta_M(P(i), \text{conf}(\bar{a}_i), \text{conf}(\bar{a}_{i+1})) \leftrightarrow (G_P)_{\text{merge}(\bar{a} \leq k+1, \bar{b} \leq k+1)} \text{ contains no } k\text{-round rules}.$$

Suppose that $\delta_M(P(i), \text{conf}(\bar{a}_i), \text{conf}(\bar{a}_{i+1}))$ holds. By the construction of $G(M, X)$, antecedents of $k+1$ rules of $(G_P)_{\text{merge}(\bar{a} \leq k, \bar{b} \leq k)}$ form $\text{conf}(\bar{a}_k)$.

In a_{k+1} -rounds, Alice must choose nodes in order to remove all such nodes in L_{N_0} . Since each such node specifies a transition rule of M , we have the claim.

Also the induction step is easily seen by the above remarks. Since the claim is Σ_0^B , it is proved by Σ_0^B -IND in V_{NK} and the claim of the lemma easily immediately follows. \square

Let G be a graph and $z_0, \dots, z_k \in V_G$. We say that $\langle z_0, \dots, z_k \rangle$ is a winning sequent for G , denoted by $WSeq(\langle z_0, \dots, z_k \rangle, G)$ if

$$G_{\langle z_0, \dots, z_{k-1} \rangle} \neq \emptyset \wedge G_{\langle z_0, \dots, z_k \rangle} = \emptyset.$$

Corollary 3 V_{NK} proves that Alice's moves for $G_A(M, X)_P$ form an accepting computation if and only if Alice wins the game:

$$\begin{aligned} \forall M, X, P \forall \bar{a} = \langle a_{0,0}, \dots, a_{s,s+1} \rangle \Big\{ (|P| = s \wedge \text{Leg}(\bar{a}) \wedge \text{Len}(\bar{a}) = (s+1)(s+2)) \rightarrow \\ (AComp(\bar{a}, P, M, X) \leftrightarrow WSeq(\text{merge}(\bar{a}, \bar{b}), G_A(M, X)_P)) \Big\}. \end{aligned}$$

(Proof). First note that Bob cannot remove any nodes in C_A unless he can move legitimately for a node in C . By Lemma 2, the only node in C_A which may remain in $(G_P)_{\text{merge}(\bar{a}, \bar{b})}$ is the acceptance node Acc . So we have

$$\text{conf}(\bar{a}_s) = C_{ACCEPT}(M, X) \leftrightarrow Acc \text{ is removed in } a_s\text{-rounds}.$$

Corollary 4 V_{NK} proves that Alice's moves for $G_R(M, X)_P$ form a rejecting computation if and only if Alice wins the game:

$$\forall M, X, P \forall \bar{a} = \langle a_{0,0}, \dots, a_{s,s+1} \rangle \left\{ (|P| = s \wedge \text{Leg}(\bar{a}) \wedge \text{Len}(\bar{a}) = (s+1)(s+2)) \rightarrow \right. \\ \left. (R\text{Comp}(\bar{a}, P, M, X) \leftrightarrow W\text{Seq}(\text{merge}(\bar{a}, \bar{b}), G_R(M, X)_P) \right\}.$$

(Proof). The proof is almost identical to Corollary 3. The only difference is if Alice moves in accordance with the computation of M on X along P then she must remove the rejecting node Rej by the last move. \square

In order to show that the strategy function yields computations of M , we need to relate Sprague-Grundy number of $G = G_A(M, X)$ or $G_R(M, X)$ and the computation of M . The next lemma asserts that Alice can always chooses options G' of $G_A(M, X)_P$ so that $sg(G') = 0$ if and only if Alice's moves form an accepting computation along P .

Lemma 4 V_{NK} proves that

$$\forall M, X, P \forall \bar{a} = \langle a_{0,0}, \dots, a_{s,s+1} \rangle \left\{ (|P| = s \wedge \text{Leg}(\bar{a})) \rightarrow \right. \\ \left. \forall k < \text{Len}(\bar{a}) (sg(G_{P * \text{merge}(\bar{a} \leq k, \bar{b} < k)}) = 0) \leftrightarrow A\text{Comp}(\bar{a}, P, M, X) \right\}$$

(Proof). Let \bar{a} be as stated. Suppose that

$$\text{conf}(\bar{a}_0) = C_{\text{INIT}}(M, X) \wedge \forall i < s \delta_M(P(i), \text{conf}(\bar{a}_i), \text{conf}(\bar{a}_{i+1})) \wedge \text{Accept}(\text{conf}(\bar{a}_s), M, X).$$

By induction on k we show that $\forall k < l_0 sg((G_P)_{\text{merge}(\bar{a} < l_0 - k, \bar{b} < l_0 - k)}) \neq 0$. If $k = 0$ then the claim follows from Corollary 3 since

$$sg(((G_P)_{\text{merge}(\bar{a} < l_0, \bar{b} < l_0)})_{z^{l_0}}) = sg((G_P)_{\text{merge}(\bar{a}, \bar{b})}) = 0.$$

For $k < l_0 - 1$, suppose by the inductive hypothesis that $sg((G_P)_{\text{merge}(\bar{a} < l_0 - k, \bar{b} < l_0 - k)}) \neq 0$. Then

$$sg(((G_P)_{\text{merge}(\bar{a} < l_0 - k, \bar{b} < l_0 - k)})_{b_{l_0 - k - 1}}) = sg((G_P)_{\text{merge}(\bar{a} < l_0 - k, \bar{b} < l_0 - k)}) \neq 0.$$

Thus by Corollary 1, we have $sg((G_P)_{\text{merge}(\bar{a} < l_0 - k, \bar{b} < l_0 - k - 1)}) \neq 0$. Since

$$sg(((G_P)_{\text{merge}(\bar{a} < l_0 - k - 1, \bar{b} < l_0 - k - 1)})_{a_{l_0 - k - 1}}) = sg((G_P)_{\text{merge}(\bar{a} < l_0 - k, \bar{b} < l_0 - k - 1)}).$$

we have $sg((G_P)_{\text{merge}(\bar{a} < l_0 - (k+1), \bar{b} < l_0 - (k+1))}) \neq 0$ as desired.

The converse direction is an immediate consequence of Corollary 2 and Corollary 3. \square

Analogously, Alice always chooses options of $G_R(M, X)_P$ whose Sprague-Grundy number is equal to 0 if and only if Bob's moves form a rejecting computation along P .

Lemma 5 V_{NK} proves that

$$\forall M, X, P \forall \bar{a} = \langle a_{0,0}, \dots, a_{s,s+1} \rangle \left\{ (|P| = s \wedge \text{Leg}(\bar{a})) \rightarrow \right. \\ \left. \forall k < \text{Len}(\bar{a}) (sg(G_R(M, X)_{P * \text{merge}(\bar{a} \leq k, \bar{b} < k)}) = 0) \leftrightarrow R\text{Comp}(\bar{a}, P, M, X) \right\}$$

(Proof). Suppose that $RComp(\bar{b}, P, M, X)$ holds. By induction on k , we show that $\forall k < (s+1)(s+2) sg(G(M, X)_{P*merge(\bar{a} \leq (s+1)(s+2)-k, \bar{b}^{(s+1)(s+2)<k})}) \neq 0$. If $k = 0$ then the claim follows from Corollary 4 since

$$sg(G(M, X)_{P*merge(\bar{a} \leq (s+1)(s+2)-k, \bar{b}^{(s+1)(s+2)<k})*b_{s,i}^Q}) = 0$$

for $i \neq 1$. The proof for $k > 0$ is identical to the one for Lemma 4. \square

Finally we show that applying the strategy function τ_A to either $G_A(M, X)_P$ or $G_R(M, X)_P$ yields either accepting or rejecting computation respectively.

Lemma 6 V_{NK} proves that if $sg(G_A(M, X)_P) \neq 0$ then the application of τ_A to $G_A(M, X)_P$ yields an accepting computation along P :

$$\begin{aligned} & \forall M, X, P \forall \bar{a} = \langle a_{0,0}, \dots, a_{s,s+1} \rangle \\ & \{(|P| = s \wedge sg(G_A(M, X)_P) \neq 0 \wedge \tau_A(\bar{b}, G_A(M, X)_P) = merge(\bar{a}, \bar{b})) \rightarrow AComp(\bar{a}, P, M, X)\} \end{aligned}$$

(Proof). Suppose that $sg(G_A(M, X)_P) \neq 0$ and let $\tau_A(\bar{b}, G_A(M, X)_P) = merge(\bar{a}, \bar{b})$. By the definition of τ_A , we have

$$\forall k \leq Len(\bar{a})(sg((G_P)_{merge(\bar{a} \leq k, \bar{b} < k)}) = 0).$$

So by Lemma 4, we have the claim. \square

Lemma 7 V_{NK} proves that if $sg(G_R(M, X)_P) \neq 0$ then the application of τ_A to $G_R(M, X)_P$ yields a rejecting computation:

$$\begin{aligned} & \forall M, X, P \forall \bar{a} = \langle a_{0,0}, \dots, a_{s,s+1} \rangle \\ & \{(|P| = s \wedge sg(G_R(M, X)_P) \neq 0 \wedge \tau_A(\bar{a}, M, X) = merge(\bar{a}, \bar{b})) \rightarrow RComp(\bar{a}, P, M, X)\} \end{aligned}$$

(Proof). Suppose that $sg(G(M, X)_P) \neq 0$. Then By Lemma 5 we have the claim by a similar argument as for Lemma 6. \square

Next lemma states that $G_A(M, X)_P$ and $G_R(M, X)_P$ play complementary roles to each other.

Lemma 8 V_{NK} proves that

$$\forall M, X, P (|P| = s \rightarrow (sg(G_A(M, X)_P) \neq 0 \leftrightarrow sg(G_A(M, X)_P) = 0)).$$

(Proof). We argue in V_{NK} . Suppose that $sg(G_A(M, X)_P) \neq 0$. We show that

$$\begin{aligned} & \forall \bar{a} (Len(\bar{a}) = (s+1)(s+2) \rightarrow \\ & \exists \bar{b} (Len(\bar{b}) \leq Len(\bar{a}) \wedge WSeq(merge(\bar{a} \leq Len(\bar{b}), \bar{b}), G_R(M, X)_P))). \quad (*) \end{aligned}$$

The proof is divided into cases. Let \bar{a} be an arbitrary list of Alice's moves with $Len(\bar{a}) = (s+1)(s+2)$ and $\bar{b}' = \langle b_{0,0}, \dots, b_{s,s} \rangle$.

If $A-Leg(\bar{a}) \wedge Comp(\bar{a})$ then by Corollary 3, we have

$$G_A(M, X)_{P*merge(\bar{a}, \bar{b}')} = \emptyset \leftrightarrow AComp(\bar{a}, M, X, P).$$

On the other hand, by Corollary 4, we have

$$G_R(M, X)_{P*merge(\bar{a}, \bar{b}')} = \emptyset \leftrightarrow RComp(\bar{a}, M, X, P).$$

Thus we have $G_R(M, X)_{P*merge(\bar{a}, \bar{b}')} \neq \emptyset$ and for any $c \in Node(G_R(M, X)_{P*merge(\bar{a}, \bar{b}')} \subseteq C_R$, we have $G_R(M, X)_{P*merge(\bar{a}, \bar{b}' * c)} = \emptyset$. Therefore we obtain $WSeq(merge(\bar{a}, \bar{b}' * c), G_R(M, X)_P)$.

If $A-Leg(\bar{a}) \wedge \neg Comp(\bar{a})$ then by Corollary 2 we have

$$G_R(M, X)_{P*merge(\bar{a}, \bar{b}')} \neq \emptyset \wedge G_R(M, X)_{P*merge(\bar{a}, \bar{b}') * c} = \emptyset.$$

Finally if $\neg A-Leg(a)$ then we can find the shortest initial part $\bar{a}' = \langle a_0, \dots, a_k \rangle$ of \bar{a} such that $A-Leg(\bar{a}') \wedge a_{k+1} \notin A_{q_{k+1}, r_{k+1}}$. Then by Lemma 1, we have x such that

$$WSeq(merge(\bar{a}', \bar{b}^{\leq k}) * a_{k+1} * x, G_R(M, X)_P).$$

Thus in any case we have $(*)$ and from this we readily have $sg(G_R(M, X)) = 0$. Conversely, if $sg(G_R(M, X)) \neq 0$ then by a similar argument, we obtain $sg(G_A(M, X)) = 0$. \square

(Proof of Lemma 2). Suppose that $sg(G_A(M, X)) \neq 0$. Then by Lemma 6, we have the first part. If $sg(G_A(M, X)) = 0$ then by Lemma 8, we have $sg(G_R(M, X)) \neq 0$ and we can apply Lemma 7

(Proof of Theorem 3). We argue in V_{NK} . Let M be an alternating Turing machine and X be an input. We define $G(M, X) = G_A(M, X)$ For other functions, we set

$$\begin{aligned} Path_A(P, M, X) &= \tau_A(P, G_A(M, X)) \\ Path_R(P, M, X) &= \tau_B(P, G_A(M, X)) \\ Comp_A(M, X, P) &= ASeq(\tau_A(\bar{b}', G_A(M, X)_P)) \\ Comp_R(M, X, P) &= ASeq(\tau_A(\bar{b}', G_A(M, X)_P)) \end{aligned}$$

where the sequence \bar{b} is defined by $\bar{b} = \langle b_{0,0}, \dots, b_{s,s} \rangle$.

The condition (1) is trivial from the definition. Conditions (2) and (3) follows from the definition of the strategy functions τ_A and τ_B .

Since we assume that $p(|X|)$ is even for all X , it follows that

$$\forall X, P (|P| = p(|X|) \rightarrow (sg(G(M, X)) = 0 \leftrightarrow sg(G_A(M, X)_P) = 0)).$$

Thus Lemma 2 implies 4. So the proof terminates. \square

Theorem 4 V_{NK} proves Σ_∞^B -IND.

(Proof). For any $\varphi(X) \in \Sigma_\infty^B$ we can construct an alternating Turing machine which decides φ in polynomial time. \square

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